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Deformations of isotropic submanifolds in Kähler manifolds *

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We study the first and second variations of isotropic submanifolds which preserve the isotropy. In order to do so, we introduce the notions of harmonic, exact and isotropic variations and investigate basic properties of isotropic submanifolds which are minimal under such deformations. Many results in this respect are then obtained. In particular, we obtain a new characterization of Maslov class in terms of such deformations.

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1. Introduction

Symplectic manifolds and their isotropic and Lagrangian submanifolds appear naturally in the context of classical mechanics and mathematical physics: The partial differential equation systems of Hamilton–Jacobi type lead to the study of isotropic and Lagrangian submanifolds and foliations in the cotangent bundle T*M of a manifold M. By Darboux's theorem and its generalizations, it is well known that the extensions of a k-manifold I to a $2n$-symplectic manifold in which I is isotropic are classified, up to a local symplectomorphism about I, by the isomorphism classes of $2(n-k)$-dimensional symplectic vector bundles over I (see, for instance, p. 24 of ref. [16]). In some sense, this says that "there is no local geometry of isotropic submanifolds", if there is no additional structure. Thus this

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article deals with the Riemannian study of isotropic submanifolds in Kähler manifolds and their deformations.

The geometry of submanifolds in Kähler geometry is quite rich: According to the behaviour of the tangent bundle of a submanifold, with respect to the action of the almost complex structure J of the ambient manifold $\tilde{M}$, there are two typical classes of submanifolds, namely, the class of complex submanifolds and the class of isotropic (or totally real [6]) submanifolds. A submanifold M of $\tilde{M}$ is a *complex submanifold* if the almost complex structure J of the ambient manifold carries each tangent space T_pM of M into itself, i.e., $J(T_pM) \subset T_pM$, for any $p \in M$ and M is an *isotropic submanifold* if J carries each tangent space T_pM into the normal space $T_p^{\perp}M$, i.e., $J(T_pM) \subset T_p^{\perp}M$, for any $p \in M$. An isotropic submanifold M in $\tilde{M}$ is *Lagrangian* if dim $\tilde{M} = 2$ dim M. In this context we can in particular study the notion of minimal isotropic submanifolds. For complex submanifolds of a Kähler manifold, it is well known that every compact complex submanifold of a Kähler manifold is minimal and stable (cf. ref. [8].) On the other hand, Lawson and Simons proved in ref. [9] that any minimal submanifold of CP^n other than complex submanifolds is unstable. In particular, their result implies every compact Lagrangian minimal submanifold in CP^m is unstable.

The stability of Lagrangian minimal submanifolds in a general Kähler manifold, especially in an Einstein–Kähler manifold, was first studied by B.Y. Chen, P.F. Leung and T. Nagano in ref. [3] (see p. 51 of ref. [2].) For example, they established in ref. [3] the second variational formula for Lagrangian minimal submanifolds in an arbitrary Kähler manifold $\tilde{M}$. By using their second variational formula, they proved that in Kähler manifolds with positive first Chern form, any stable minimal Lagrangian submanifold L has vanishing first cohomology group, i.e., $H^1(L; R) = 0$. Moreover, if $\tilde{M}$ is a Kähler manifold with nonpositive first Chern form, then any minimal Lagrangian submanifold L of $\tilde{M}$ is unstable (see also refs. [7,11]).

Inspired by Wirtinger's inequality, Oh introduced in ref. [11] the notion of Hamiltonian deformations for Lagrangian submanifolds. By applying the second variational formula of Chen, Leung and Nagano, he investigated in ref. [11] Hamiltonian stability for Lagrangian submanifolds. Recently, Oh generalized Chen, Leung and Nagano's formula to more general Lagrangian submanifolds and obtained several applications (cf. ref. [12] for details).

In this article, we investigate this problem in a more general setting. Namely, we study the first and the second variations of isotropic submanifolds which preserve the isotropy. For this purpose we generalize the notion of Hamiltonian variations of Lagrangian submanifolds to the notions of isotropic variations and exact variations for a general isotropic submanifold in a Kähler manifold. We also introduce the notions of harmonic variations and harmonic minimal submanifolds. We will investigate basic properties of isotropic submanifolds which are minimal under isotropic deformations, exact deformations, and harmonic variations. Many results in this respect are then obtained.

As an application we deduce easily an interesting characterization of the Maslov class in terms of deformation. Remember that the Maslov class appears in the resolution of a Hamilton–Jacobi system, as an obstruction to the transversality of a Lagrangian submanifold of a cotangent bundle, with the vertical subbundle. In ref. [10] the second author could express this class in terms of the mean curvature vector of the Lagrangian submanifold (see ref. [15] for its generalizations). Here, our framework enables us to solve a problem of Le Khong Van and A.T. Fomenko [14]: We prove in section 3 that a compact Lagrangian submanifold L in an Einstein–Kähler manifold is harmonic minimal if and only if the Maslov class of L vanishes. In particular, this shows that the harmonic minimality of a closed curve in $\mathbb{C}^1$ is equivalent to the condition of vanishing rotation index. In section 3, we also study the curvature of exact minimal (or E-minimal) isotropic surfaces in $\mathbb{C}^2$. For such a surface, we obtain a relationship between its Gauss curvature and its mean curvature. Moreover, we completely classify such surfaces with constant mean curvature.

In section 4, we derive the second variational formulas for isotropic submanifolds in an arbitrary Kähler manifold, either under the usual deformations or under the new deformations. By applying these second variational formulas, we study the stability of isotropic submanifolds under either usual or the new deformations. Several results in this direction are then obtained. For example, we prove that every compact hypersurface of a real projective $(n+1)$-space RP^{n+1}, embedded standardly in CP^{n+1}, is an isotropic unstable E-minimal submanifold of CP^{n+1}, although RP^{n+1} itself is a Hamiltonian stable Lagrangian submanifold of CP^{n+1}. Of course, our situation is more complicated than the Lagrangian case, since for an isotropic submanifold which is not Lagrangian the normal bundle contains a complex subbundle on which we have very little information.

2. Isotropic submanifolds

Let M be an m-dimensional Riemannian manifold with Levi-Civita connection $\tilde{\nabla}$ and let $i: I \to \tilde{M}$ be an isometric immersion of an n-dimensional compact Riemannian manifold I into a Riemannian manifold $\tilde{M}$. Denote by $\langle\ ,\ \rangle$ the Riemannian metric on I as well as on $\tilde{M}$. Let ∇ and R (respectively, $\tilde{\nabla}$ and $\tilde{R}$) be the Levi-Civita connection and the Riemann curvature tensor on I (respectively, on $\tilde{M}$).

Denote by

$$\sigma: TI \times TI \to T^{\perp} I$$

the second fundamental form of the immersion and by A the shape operator of the immersion. Then A satisfies

$$\langle \sigma(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle , \quad \forall X, Y \in TI, \xi \in T^\perp I .$$

With respect to $\langle \ , \ \rangle$, the shape operator can also be regarded as a bilinear map $A: TI \times T^\perp I \to TI$.

The mean curvature vector field of the immersion is given by $H = n^{-1}$ Tr σ ($n = \dim I$). Finally, we denote by $\nabla^\perp$ and $R^\perp$ the normal connection and the normal curvature tensor of the immersion. We have the following formula of Weingarten:

$$\tilde{\nabla}_X \xi = \nabla^\perp_X \xi - A_\xi X , \quad \forall X \in TI, \xi \in T^\perp I .$$

Suppose $\tilde{M}$ is a Kähler manifold. Denote by J the almost complex structure on $\tilde{M}$ and by Ω the symplectic structure defined by

$$\Omega(X, Y) = \langle JX, Y \rangle , \quad \forall X, Y \in T\tilde{M} .$$

From the definition of Ω, it is clear that an immersion $i: I \to \tilde{M}$ is an isotropic immersion if and only if $i^*\Omega = 0$. An isotropic submanifold I is said to be *Lagrangian* if $\dim_C \tilde{M} = \dim_R I$.

When I is an isotropic submanifold, the normal bundle of I has the following canonical orthogonal decomposition:

$$T^\perp I = J(TI) \oplus \nu , \tag{2.1}$$

where ν is the maximal complex subbundle of the normal bundle. From the definition, it is clear that an isotropic submanifold I is Lagrangian if and only if the maximal complex normal subbundle $\nu = \{0\}$.

If I is an isotropic submanifold and ξ a cross-section of the normal bundle $T^\perp I$, i.e., $\xi \in \Gamma(T^\perp I)$, then there is a one-form α_ξ on I associated with ξ defined by

$$\alpha_\xi(X) = \Omega(\xi, X) = \langle J\xi, X \rangle , \quad \forall X \in TI . \tag{2.2}$$

Conversely, if β is a one-form on the isotropic submanifold I and $\beta^\#$ the tangent vector field on I dual to β with respect to the metric defined on I, then $\alpha_{J\beta^\#} = -\beta$.

We need the following general lemma.

Lemma 2.1. *Let I be an isotropic submanifold of a Kähler manifold $\tilde{M}$ and ξ a normal vector field of I in $\tilde{M}$. Then the one-form α_ξ is closed, i.e., $d\alpha_\xi = 0$, if and only if*

$$\langle \nabla^\perp_X \xi, JY \rangle = \langle \nabla^\perp_Y \xi, JX \rangle ,$$

for any vector fields X, Y tangent to I.

Proof. From (2.2) we have

$$d\alpha_\xi=0\Leftrightarrow 0=X(\alpha_\xi(Y))-X(\alpha_\xi(X))-\alpha_\xi([X,Y])$$
$$=\langle\tilde{\nabla}_X J\xi, Y\rangle-\langle\tilde{\nabla}_Y J\xi, X\rangle=\langle\tilde{\nabla}_Y\xi, JX\rangle-\langle\tilde{\nabla}_X, JY\rangle$$
$$=\langle\nabla^\perp_Y\xi, JX\rangle-\langle\nabla^\perp_X\xi, JY\rangle\ . \qquad\square$$

Let I be an n-dimensional isotropic submanifold of a Kählerian manifold $\tilde{M}$ of complex dimension $n+r$. We choose a local field of orthonormal frames

$$e_1,\ldots,e_n,e_{n+1},\ldots,e_{n+r}\,,$$
$$e_{1^*}=Je_1\,,\quad\ldots,\quad e_{n^*}=Je_n\,,\quad e_{(n+1)^*}=Je_{n+1}\,,\quad\ldots,\quad e_{(n+r)^*}=Je_{n+r}\,,$$

in $\tilde{M}$ in such a way that, restricted to I, $e_1,\ldots,e_n$ are tangent to I. With respect to the frame field of $\tilde{M}$ chosen above, let

$$\omega^1,\ldots,\omega^n,\omega^{n+1},\ldots,\omega^{n+r},\omega^{1^*},\ldots,\omega^{n^*},\omega^{(n+1)^*},\ldots,\omega^{(n+r)^*}$$

be the field of dual frames.

We use the following convention on the range of indices unless otherwise stated:

$$A,B,C,D=1,\ldots,n+r,1^*,\ldots,(n+r)^*\,;\qquad i,j,k,l=1,\ldots,n\,;$$
$$\lambda,\mu=n+1,\ldots,n+r\,;\qquad \beta,\gamma,\delta=n+1,\ldots,n+r,1^*,\ldots,(n+r)^*\,.$$

The structure equations of $\tilde{M}$ are given by

$$d\omega^A=-\sum\omega^A_B\wedge\omega^B\,,\qquad \omega^A_B+\omega^B_A=0\,,$$
$$\omega^i_j+\omega^j_i=0\,,\qquad \omega^i_j=\omega^{i^*}_{j^*}\,,\qquad \omega^{i^*}_j=\omega^{j^*}_i\,,\tag{2.3}$$
$$\omega^\lambda_\mu+\omega^\mu_\lambda=0\,,\qquad \omega^\lambda_\mu=\omega^{\lambda^*}_{\mu^*}\,,\qquad \omega^{\lambda^*}_\mu=\omega^{\mu^*}_\lambda\,,$$
$$\omega^i_\lambda+\omega^\lambda_i=0\,,\qquad \omega^i_\lambda=\omega^{i^*}_{\lambda^*}\,,\qquad \omega^{i^*}_\lambda=\omega^{\lambda^*}_i\,;\tag{2.4}$$

$$d\omega^A_B=-\sum\omega^A_C\wedge\omega^C_B+\tilde{\Omega}^A_B\,,\qquad \tilde{\Omega}^A_B=\frac{1}{2}\sum\tilde{R}^A_{BCD}\omega^C\wedge\omega^D\,.\tag{2.5}$$

Restricting these forms to I, we have the structure equations of the immersion:

$$\omega^\beta=0\,,\tag{2.6}$$
$$d\omega^i=-\sum\omega^i_k\wedge\omega^k\,,\tag{2.7}$$
$$d\omega^i_j=-\sum\omega^i_k\wedge\omega^k_j+\Omega^i_j\,,\qquad \Omega^i_j=\frac{1}{2}\sum R^i_{jkl}\omega^k\wedge\omega^l\,,\tag{2.8}$$
$$\Omega^i_j=\tilde{\Omega}^i_j-\sum\omega^i_\beta\wedge\omega^\beta_j\,,\tag{2.9}$$
$$d\omega^\beta_\gamma=-\sum\omega^\beta_\delta\wedge\omega^\delta_\gamma+\Omega^\beta_\gamma\,,\qquad \Omega^\beta_\gamma=\frac{1}{2}\sum R^\beta_{\gamma kl}\omega^k\wedge\omega^l\,,\tag{2.10}$$

$$\Omega_\gamma^\beta = \tilde{\Omega}_\gamma^\beta - \sum \omega_i^\beta \wedge \omega_\gamma^i . \tag{2.11}$$

For the isotropic submanifold I in $\tilde{M}$, let Θ be the one-form on I given by

$$\Theta = \sum_{i=1}^{n} \omega_i^{i^*} . \tag{2.12}$$

Then we have

$$\Theta(e_j) = \sum_i \langle \sigma(e_i, e_j), e_{i^*} \rangle = \sum_i \langle \sigma(e_i, e_i), e_{j^*} \rangle = -n\alpha_H(e_j) . \tag{2.13}$$

Thus we obtain

$$\alpha_H = -(1/n)\Theta . \tag{2.14}$$

We give the following result for later use.

Lemma 2.2. *If I is an isotropic submanifold of a Kähler manifold $\tilde{M}$, then*

$$d\Theta = 2 \sum_{\mu,j,k} (h_{ij}^\mu h_{ik}^{\mu^*} - h_{ik}^\mu h_{ij}^{\mu^*}) \omega^j \wedge \omega^k + \sum_i \tilde{\Omega}_i^{i^*} , \tag{2.15}$$

$$\sum_i \tilde{\Omega}_i^{i^*} = \frac{1}{2} \sum_{j,k} \left(\tilde{S}_{jk^*} + \sum_\mu \tilde{R}_{\mu jk}^{\mu^*} \right) \omega^k \wedge \omega^j , \tag{2.16}$$

where (h_{ij}^β) denote the coefficients of the second fundamental form σ, and $\tilde{S}$ and $\tilde{R}$ are the Ricci tensor and the Riemann curvature tensor of $\tilde{M}$, respectively. In particular, if $\tilde{M}$ is Einsteinian, we have

$$d\Theta = 2 \sum_{\mu,j,k} (h_{ij}^\mu h_{ik}^{\mu^*} - h_{ik}^\mu h_{ij}^{\mu^*}) \omega^j \wedge \omega^k + \frac{1}{2} \sum_{j,k} \sum_\mu \tilde{R}_{\mu jk}^{\mu^*} \omega^k \wedge \omega^j . \tag{2.17}$$

Proof. From (2.4) and (2.5) we have

$$\begin{aligned} d\Theta = & \sum \omega_j^{i^*} \wedge \omega_i^j + \sum \omega_{j^*}^{i^*} \wedge \omega_i^{j^*} \\ & + \sum \omega_\mu^{i^*} \wedge \omega_i^\mu + \sum \omega_{\mu^*}^{i^*} \wedge \omega_i^{\mu^*} + \sum \tilde{\Omega}_i^{i^*} \\ = & 2 \sum \omega_i^\mu \wedge \omega_i^{\mu^*} + \sum \tilde{\Omega}_i^{i^*} . \end{aligned} \tag{2.18}$$

On the other hand,

$$\sum \tilde{\Omega}_i^{i^*} = \frac{1}{2} \sum \tilde{R}_{ijk}^{i^*} \omega^j \wedge \omega^k , \tag{2.19}$$

$$\begin{aligned} \sum_i \tilde{R}_{ijk}^{i^*} & = \sum \tilde{R}_{ii^*k}^j + \sum \tilde{R}_{jik}^{i^*} = \sum \tilde{R}_{jik^*}^i + \sum \tilde{R}_{ji^*k^*}^{i^*} \\ & = -\tilde{S}_{jk^*} - \sum_\mu \tilde{R}_{j\mu k^*}^\mu - \sum_\mu \tilde{R}_{j\mu^*k^*}^{\mu^*} \\ & = -\tilde{S}_{jk^*} - \sum_\mu \tilde{R}_{\mu jk}^{\mu^*} . \end{aligned} \tag{2.20}$$

In particular, if $\tilde{M}$ is Einsteinian, the isotropy of I implies $\tilde{S}_{jk*}=0$. Combining this with (2.19) and (2.20) we obtain the lemma. □

Lemma 2.2 can be regarded as a generalization of ref. [10].

As a consequence of lemma 2.2 we have the following relation between α_H and the first Chern form $\gamma_1(\nu)$ of the maximal complex normal subbundle ν.

Proposition 2.3. *If I is an isotropic submanifold of a Kählerian manifold $\tilde{M}$ of constant holomorphic sectional curvature, then*

$$d\alpha_H=(16\pi/n)\gamma_1(\nu)\ , \tag{2.21}$$

where

$$\gamma_1(\nu)=\frac{1}{4\pi}\sum \Omega_{\mu}^{\mu^*} \tag{2.22}$$

is the first Chern form of the maximal complex normal subbundle ν of the normal bundle $T^{\perp}I$.

Proof. If $\tilde{M}$ is a Kählerian manifold of constant holomorphic sectional curvature c, then the Riemann curvature tensor $\tilde{R}$ of $\tilde{M}$ is given by

$$\begin{aligned}\tilde{R}(X,Y)Z=\tfrac{1}{4}c\{&\langle Y,Z\rangle X-\langle X,Z\rangle Y+\langle JY,Z\rangle JX\\ &-\langle JX,Z\rangle JY+2\langle X,JY\rangle JZ\}\ .\end{aligned}$$

Thus we have

$$\tilde{R}^{\mu^*}_{\mu jk}=0\ .$$

Moreover, since $\tilde{M}$ is Einstein–Kähler and I is isotropic, we also have $\tilde{S}_{jk*}=0$. Therefore from (2.11) and lemma 2.2 we have

$$\begin{aligned}d\Theta&=2\sum_i \omega_i^{\mu}\wedge\omega_i^{\mu^*}=2\sum_{\mu,j,k}(h^{\mu}_{ij}h^{\mu^*}_{ik}-h^{\mu}_{ik}h^{\mu^*}_{ij})\omega^j\wedge\omega^k\\ &=-2\sum_{\mu,j,k}R^{\mu^*}_{\mu jk}\omega^j\wedge\omega^k=-4\sum\Omega^{\mu^*}_{\mu}\ .\end{aligned}$$

This implies the proposition. □

Proposition 2.3 yields the following

Corollary 2.4 *Let I be an isotropic submanifold of a Kähler manifold $\tilde{M}$ of constant holomorphic sectional curvature. Then*

(1) the first Chern class of the maximal complex normal subbundle ν vanishes, i.e., $c_1(\nu)=[\gamma_1(\nu)]=0$.

(2) the one-form α_H is closed if and only if the first Chern form vanishes, i.e., $\gamma_1(\nu)=0$.

Combining lemma 2.1 and corollary 2.4, we obtain

Corollary 2.5. *Let I be an isotropic submanifold of a Kähler manifold $\tilde{M}$ of constant holomorphic sectional curvature. If I has parallel mean curvature vector field, then the maximal complex normal subbundle ν has vanishing first Chern form, i.e., $\gamma_1(\nu)=0$.*

3. Deformations

3.1. DEFORMATIONS

Let I be a compact submanifold of a Riemannian manifold $\tilde{M}$ with boundary ∂I (∂I may be empty). Suppose ξ is a vector field on $\tilde{M}$ defined on a neighbourhood of I such that $\xi|_I$ is normal to I and $\xi|_{\partial I}\equiv 0$. Let ϕ_t denote the flow generated by ξ in a neighbourhood of I in $\tilde{M}$. The flow $\{\phi_t\}$ gives rise to a normal variation of I in $\tilde{M}$, which is denoted either by ξ or by $\{\phi_t\}$. In this case, the normal vector field $\xi|_I$ is called the *variation vector field* of the normal variation.

Let $\mathscr{V}(t)=\text{volume}(\phi_t(I))$, the volume of $\phi_t(I)$ via ξ. By $\mathscr{V}'(\xi)$ and $\mathscr{V}''(\xi)$ we mean the values of the first and the second derivatives of $\mathscr{V}(t)$, respectively, with respect to t, evaluated at $t=0$. It is well known that (see, for instance, ref. [8])

$$\mathscr{V}'(\xi)=-n\int_I \langle H,\xi\rangle \,\mathrm{d}v\,. \tag{3.1}$$

For our purpose we give the following expression of the second variational formula for later use (see also ref. [8]).

Lemma 3.1. *Let I be a compact n-dimensional submanifold of a Riemannian manifold $\tilde{M}$. Suppose that ξ is a vector field on $\tilde{M}$ such that $\xi|_I$ is normal to I. Let i_t denote the flow generated by ξ in a neighbourhood of I in $\tilde{M}$. Then we have*

$$\mathscr{V}''(\xi)=\int_I \{\|\nabla^\perp\xi\|^2-\|A_\xi\|^2+n^2\langle H,\xi\rangle^2-\tilde{S}(\xi,\xi)-n\langle H,\tilde{\nabla}_\xi\xi\rangle\}\,\mathrm{d}v\,, \tag{3.2}$$

where $\tilde{S}(\xi,\eta)=\sum_{i=1}^n \tilde{R}(\xi,e_i,e_i,\eta)$, $(e_1,\dots,e_n)$ is a local field of orthonormal frames of TI, and $\tilde{R}$ is the Riemann curvature tensor of the ambient manifold $\tilde{M}$.

Proof. Let us choose a local field of orthonormal frames $e_1,\dots,e_n$ of TI at $t=0$ and extend it to a neighbourhood of I by ϕ_t. We shall denote this local frame field also by $e_1,\dots,e_n$. Then the volume element of $\phi_t(M)$ is given by

$$\sqrt{\langle e_1 \wedge \cdots \wedge e_n, e_1 \wedge \cdots \wedge e_n \rangle} * 1 ,$$

where $*1$ is the volume element of I at $t=0$. Thus we have

$$\mathscr{V}(t) = \int_I \sqrt{\langle e_1 \wedge \cdots \wedge e_n, e_1 \wedge \cdots \wedge e_n \rangle} * 1 .$$

Therefore

$$\mathscr{V}''(\xi) = \int_I \xi(\xi(\sqrt{\langle e_1 \wedge \cdots \wedge e_n, e_1 \wedge \cdots \wedge e_n \rangle})) * 1 . \tag{3.3}$$

By direct computation we have at $t=0$ the following:

$$\begin{aligned}
&\xi(\xi(\sqrt{\langle e_1 \wedge \cdots \wedge e_n, e_1 \wedge \cdots \wedge e_n \rangle}))\\
&\quad = \xi\Big(\sum_i \langle e_1 \wedge \cdots \wedge \tilde{\nabla}_\xi e_i \wedge \cdots \wedge e_n, e_1 \wedge \cdots \wedge e_n \rangle \Big)\\
&\qquad - \sum_i \langle e_1 \wedge \cdots \wedge \tilde{\nabla}_\xi e_i \wedge \cdots \wedge e_n, e_1 \wedge \cdots \wedge e_n \rangle^2\\
&\quad = \sum_{i \neq j} \langle e_1 \wedge \cdots \wedge \tilde{\nabla}_\xi e_i \wedge \cdots \wedge \tilde{\nabla}_\xi e_j \wedge \cdots \wedge e_n, e_1 \wedge \cdots \wedge e_n \rangle\\
&\qquad + \sum_i \langle e_1 \wedge \cdots \wedge \tilde{\nabla}_\xi \tilde{\nabla}_\xi e_i \wedge \cdots \wedge e_n, e_1 \wedge \cdots \wedge e_n \rangle\\
&\qquad + \sum_{i,j} \langle e_1 \wedge \cdots \wedge \tilde{\nabla}_\xi e_i \wedge \cdots \wedge e_n, e_1 \wedge \cdots \wedge \tilde{\nabla}_\xi e_j \wedge \cdots \wedge e_n \rangle\\
&\qquad - \sum_i \langle e_1 \wedge \cdots \wedge \tilde{\nabla}_\xi e_i \wedge \cdots \wedge e_n, e_1 \wedge \cdots \wedge e_n \rangle^2\\
&\quad = \sum_{i,j} \{\langle A_\xi e_i, e_i \rangle \langle A_\xi e_j, e_j \rangle - \langle A_\xi e_i, e_j \rangle^2\} + \sum_i \langle \tilde{R}_{\xi, e_i} \xi, e_i \rangle\\
&\qquad + \operatorname{div}(\tilde{\nabla}_\xi \xi)^{\mathrm{T}} - \sum_i \langle A_{(\tilde{\nabla}_\xi \xi)^\perp} e_i, e_i \rangle\\
&\qquad + \Big(\sum_i \langle A_\xi e_i, e_i \rangle \Big)^2 + |\nabla^\perp \xi|^2 - n^2 \langle \xi, H \rangle^2 .
\end{aligned}$$

From this together with (3.3) and the divergence theorem we obtain the lemma. □

3.2. ISOTROPIC DEFORMATIONS

If $i: I \to \tilde{M}$ is an isotropic isometric immersion from I into a Kähler manifold $\tilde{M}$, we may only consider normal variations ξ such that $\phi_t(I)$ stays isotropic for each sufficiently small t. Then the infinitesimal version of this condition is given by

$$i^*(\mathscr{L}_\xi\Omega)=0\ ,$$

where $\mathscr{L}$ denotes the Lie derivative (see, e.g., ref. [1]). We give the following general lemma.

Lemma 3.2. *$i^*(\mathscr{L}_\xi\Omega)=0 \Leftrightarrow \alpha_\xi$ is a closed one-form on I, where α_ξ is the one-form defined by (2.2).*

Proof. Let ι and d denote the interior product and the exterior differential operator, respectively. Then we have

$$\mathscr{L}=d\circ\iota+\iota\circ d\ .$$

Hence

$$i^*(\mathscr{L}_\xi\Omega)=0 \Leftrightarrow i^*(d(\iota_\xi\Omega))=0 \Leftrightarrow d\alpha_\xi=0\ . \qquad \square$$

Lemma 3.2 leads us to give the following definition for isotropic submanifolds (cf. ref. [11] for the Lagrangian case):

Definition 3.3. Let I be an isotropic submanifold of a Kähler manifold $\tilde{M}$. Then a vector field ξ along I is called *isotropic* (respectively, *exact* or *harmonic*) if the one-form α_ξ is closed (respectively, exact or harmonic). A normal variation $\{\phi_t\}$ of I in $\tilde{M}$ is called *isotropic* (respectively, *exact* or *harmonic*) if the variation vector field along I of $\{\phi_t\}$ is isotropic (respectively, exact or harmonic).

For a compact Lagrangian submanifold L, harmonic vector fields and harmonic variations are defined when $H^1(L; R)\neq\{0\}$. In this case, it follows from definition 3.3 that the zero vector field is the only normal vector field which is exact and harmonic at the same time.

For each normal vector field ξ along the isotropic submanifold I in a Kähler manifold $\tilde{M}$, there is a unique normal variation $\{\phi_t\}$ defined by $\phi_t(m)=\exp_{i(m)}(t\xi)$, $m\in I$. This normal variation generates a vector field of $\tilde{M}$ which is an extension of ξ defined on a neighbourhood of $i(I)$. We shall denote this extended vector field also by ξ. For simplicity, we identify the unique normal variation associated with ξ with the normal vector field ξ. It is obvious that we have $\tilde{\nabla}_\xi\xi=0$ for any such normal variation ξ.

For an isotropic submanifold I in $\tilde{M}$, we denote by $\mathscr{I}$, $\mathscr{E}$ and $\mathscr{H}$ the sets of all isotropic, exact and harmonic normal variations ξ on I, obtained from the corresponding normal vector fields ξ. It is easy to see that $\mathscr{I}\supset\mathscr{E}$ and $\mathscr{I}\supset\mathscr{H}$. Moreover, if the normal vector field ξ is in $\Gamma(\nu)$, then the corresponding normal variation ξ is automatically isotropic, exact and also harmonic.

Of course, if $H^1(I, R)=\{0\}$, the notions of isotropic deformations, exact deformations and harmonic deformations are equivalent.

Let $\wedge^p(I)$ denote the space of all p-forms on I. Denote by $\mathscr{Z}^p(I)$ and $\mathscr{E}^p(I)$ the subspaces of $\wedge^p(I)$ consisting of all closed p-forms and all exact p-forms on I, respectively.

For any two p-forms ω, $\psi \in \wedge^p(I)$, we define a (global) scalar product of ω, ψ by

$$(\omega, \psi) = \int_I \omega \wedge *\psi, \tag{3.4}$$

whenever it is defined, where $*$ is the Hodge star operator.

Definition 3.4. An isotropic submanifold I of a Kähler manifold is said to be *isotropic minimal* (respectively, *E-minimal* or *harmonic minimal*) if $\mathscr{V}'(\xi) = 0$ for any isotropic (respectively, exact or harmonic) variation ξ of I, i.e., we have $\mathscr{V}'(\xi) = 0$ for any ξ in $\mathscr{I}$ (respectively, for any ξ in $\mathscr{E}$ or $\mathscr{H}$). The isotropic submanifold I is said to be *stable* (in the usual sense) (respectively, *isotropic stable*, *E-stable* or *harmonic stable*) if $\mathscr{V}''(\xi) \geqslant 0$ for every normal variation (respectively, isotropic variation, exact variation or harmonic variation) of I. Otherwise, I is said to be *unstable* (in the usual sense) (respectively, *isotropic unstable*, *E-unstable* or *harmonic unstable*).

Clearly, every minimal isotropic submanifolds in a Kähler manifold is automatically isotropic minimal, E-minimal, and harmonic minimal. Moreover, there do exist isotropic minimal, E-minimal and harmonic minimal isotropic submanifolds which are not minimal in the usual sense (see examples below).

For isotropic submanifolds we have the following

Proposition 3.5. *Let I be a compact isotropic submanifold of a Kähler manifold $\tilde{M}$. Then*

(i) I is isotropic minimal if and only if $H \in \Gamma(J(TI))$ and $\alpha_H \in \mathscr{Z}^1(I)^{\perp} = \{\delta\phi : \phi \in \wedge^2 I\}$.

(ii) I is E-minimal if and only if $H \in \Gamma(J(TI))$ and $\delta\alpha_H = 0$, where δ is the codifferential operator on I.

(iii) I is harmonic minimal if and only if $H \in \Gamma(J(TI))$ and α_H is the sum of an exact one-form and a co-exact one-form.

Proof.

(i) I is isotropic minimal if and only if

$$\mathscr{V}'(\xi) = -n \int_I \langle \xi, H \rangle \, \mathrm{d}v = 0 \tag{3.5}$$

for every isotropic variation ξ on I. Since for each $\xi\in\Gamma(\nu)$ the normal variation ξ is isotropic, (3.5) implies $H\in\Gamma(J(TI))$. Hence, if I is isotropic minimal, then we have

$$\mathscr{V}'(\xi)=-n\int_I \langle\xi,H\rangle \,\mathrm{d}v=-n(\alpha_\xi,\alpha_H)=0\,.$$

Thus, by combining this with (3.4), we obtain $\alpha_H\in\mathscr{Z}^1(I)^\perp$. Furthermore, for a compact Riemannian manifold, $\mathscr{Z}^1(I)^\perp=\{\delta\phi:\phi\in\wedge^2 I\}$. The converse of this is clear.

(ii) From (i) it follows that an isotropic submanifold I is E-minimal if and only if $H\in\Gamma(J(TI))$ and

$$(\alpha_\xi,\alpha_H)=0 \tag{3.6}$$

for exact variations $\xi\in\mathscr{H}$. Since a normal variation ξ is exact if and only if $\alpha_\xi=df$ for some function f on I, we have

$$(df,\alpha_H)=(f,\delta\alpha_H)=0\,,\quad f\in C^\infty(I)\,.$$

This is equivalent to saying that $\delta\alpha_H=0$.

(iii) Formula (3.1) implies that I is harmonic minimal if and only if α_H is perpendicular to harmonic one-forms on I. Thus, by the Hodge–de Rham decomposition of one-forms, we conclude to statement (iii). □

From proposition 3.5 we have the following

Corollary 3.6. *Let I be a compact isotropic submanifold of a Kähler manifold $\tilde{M}$. If $d\alpha_H=0$, then*

(i) I is isotropic minimal if and only if L is minimal in the usual sense;

(ii) I is E-minimal if and only if α_H is a harmonic one-form, i.e., $\Delta\alpha_H=0$;

(iii) I is harmonic minimal if and only if α_H is an exact one-form.

Proof.

(i) Let I be an isotropic submanifold of a Kähler manifold $\tilde{M}$. Assume $\mathrm{d}\alpha_H=0$. If I is isotropic minimal, then $\alpha_H\in\mathscr{Z}^1(I)^\perp$. Thus, $(\alpha_H,\alpha_H)=0$. This implies $\alpha_H=0$, which is equivalent to saying that L is minimal in the usual sense. The converse of this is trivial.

(ii) If I is E-minimal, proposition 3.5 yields $\delta\alpha_H=0$. Since $d\alpha_H=0$, α_H is a harmonic one-form.

(iii) If I is harmonic minimal, proposition 3.5 implies that α_H is the sum of an exact one-form and a coexact one-form. On the other hand, the condition $d\alpha_H=0$ implies that α_H is orthogonal to every coexact one-form on L. Therefore, α_H must be exact.

The converse is clear. □

Lemma 2.2 and corollary 3.6 imply the following

Corollary 3.7. *Let L be a compact Lagrangian submanifold of an Einstein–Kähler manifold* $\tilde{M}$. *Then*

(i) L is isotropic minimal if and only if L is minimal in the usual sense;
(ii) L is E-minimal if and only if α_H *is a harmonic one-form, i.e.,* $\Delta\alpha_H = 0$;
(iii) L is harmonic minimal if and only if α_H *is an exact one-form.*

3.3. EXAMPLES OF E-MINIMAL ISOTROPIC SUBMANIFOLDS

(i) The standard torus T^n in $\mathbb{C}^{n+p}$ is a compact isotropic submanifold which is E-minimal, but not minimal in the usual sense and also not harmonic minimal.

(ii) Any isotropic submanifold of a Kähler manifold with parallel mean curvature vector is E-minimal (cf. lemma 2.1).

(iii) An easy construction of isotropic submanifolds which are E-minimal is the following: Take a submanifold I with parallel mean curvature vector field of a totally geodesic Lagrangian (or isotropic) submanifold of a Kähler manifold $\tilde{M}$. Then I is an E-minimal isotropic submanifold of $\tilde{M}$ since it is an obvious consequence of the following

Lemma 3.8. *Let* I *be an isotropic submanifold of a Kähler manifold* $\tilde{M}$ *with* $H \in \Gamma(J(TI))$. *Then* $\delta\alpha_H = 0$ *if and only if*

$$\sum_i \langle \nabla^\perp_{e_i} H, Je_i \rangle = 0 .$$

Proof. If $H \in \Gamma(J(TI))$, then

$$\delta\alpha_H = 0 \Leftrightarrow \sum_i \langle \nabla_{e_i}(JH), e_i \rangle = 0$$

$$\Leftrightarrow \sum_i \langle \tilde{\nabla}_{e_i} JH, e_i \rangle = 0$$

$$\Leftrightarrow \sum_i \langle \nabla^\perp_{e_i} H, Je_i \rangle = 0 . \qquad \square$$

3.4. EXAMPLES AND RESULTS ON HARMONIC MINIMAL SUBMANIFOLDS

First we make the following observations.

Observation 3.9. Statement (iii) of corollary 3.7 says that a compact Lagrangian submanifold L in an Einstein–Kähler manifold is harmonic minimal if and only if the Maslov class of L vanishes, i.e., $[\alpha_H]=0$. From this we conclude that there exist many harmonic minimal Lagrangian submanifolds.

Observation 3.10. If a curve I in a complex Euclidean m-space is harmonic minimal and not minimal in the usual sense, then the image of the curve is contained in a complex line of $\mathbb{C}^m$.

This fact can be seen as follows: Let e_1 be a unit tangent vector field of the curve. Since the curve is harmonic minimal, proposition 3.5 implies that the mean curvature vector H of the curve satisfies $H=\kappa Je_1$ for some function κ on I. Thus $JH=-\kappa e_1$. Hence, by using the fact that $\mathbb{C}^m$ is Kählerian, we may obtain $\tilde{\nabla}_{e_1}(e_1 \wedge Je_1)=0$. This implies that the curve I is contained in a complex line of $\mathbb{C}^m$.

Observation 3.11. If a closed curve in a complex Euclidean m-space $\mathbb{C}^m$ is E-minimal, then the curve is a circle which lies in a complex line $\mathbb{C}^1$ of $\mathbb{C}^m$.

This can be seen as follows: If a closed curve is E-minimal, then, by proposition 3.5, the mean curvature vector H is nonzero and $H=\kappa Je_1$ for some function κ. Thus, by using the same argument in as observation 3.10, we may conclude that the curve lies in a complex line $\mathbb{C}^1$ of $\mathbb{C}^m$. Now, since the curve is E-minimal, the one-form α_H is co-closed by proposition 3.5. Because the curve is one-dimensional, this implies that the mean curvature has constant length. In other words, the plane curve in $\mathbb{C}^1$ has constant curvature. Hence, it is a circle in $\mathbb{C}^1$.

If L is a curve in a real two-dimensional Kähler manifold $\tilde{M}$ and e_1 is a global unit tangent vector field of the curve, then the mean curvature vector H of the curve satisfies $H=\kappa Je_1$ for some function κ defined on the whole curve. We call this function the *curvature* of the curve.

The following theorem provides a geometric characterization of harmonic minimal Lagrangian curves.

Theorem 3.12. *Let L be a closed curve in a real two-dimensional Kähler manifold $\tilde{M}$. Then L is harmonic minimal if and only if L has zero total curvature, i.e.,* $\int_L \kappa \, ds=0$, *where s is an arc parametrization of L.*

Proof. If L is harmonic minimal, then proposition 3.4 implies that the one-form α_H is exact, since L is one-dimensional. Thus, $\alpha_H=df=f'(s)\, ds$. From the definition of α_H we have $f'(s)=-\kappa$. Since L is closed, we obtain $\int_L \kappa \, ds=0$.

Conversely, if $\int_L \kappa \, ds=0$, then there exists a function $f(s)$ on L such that $f'(s)=\kappa(s)$. This implies that the one-form $\alpha_H=df$ is exact. Thus, by applying proposition 3.4, the curve is harmonic minimal. □

If the ambient space $\tilde{M}$ is the complex plane $\mathbb{C}^1$, theorem 3.12 implies immediately the following relation between rotation index and minimal harmonic closed curves in $\mathbb{C}^1$. (A similar result is well known for a closed curve in $\mathbb{C}^1$ with null Maslov class.)

Corollary 3.13. *Let L be a closed curve in the complex plane $\mathbb{C}^1$. Then L is harmonic minimal if and only if the rotation index of L is zero.*

Proof. This follows from theorem 3.12 and the fact that the rotation index of a closed plane curve L is given by $(2\pi)^{-1}\int_L \kappa\, ds$. □

By combining theorem 3.12 and the Gauss–Bonnet theorem we have the following

Theorem 3.14. *Let $\tilde{M}$ be a real two-dimensional Kähler manifold and L an embedded closed curve L in $\tilde{M}$ which bounds a simply connected region D of $\tilde{M}$. Then L is harmonic minimal if and only if the region D of $\tilde{M}$ has total Gauss curvature 2π, i.e., $\int_D G\, dv = 2\pi$.*

Proof. Follows directly from theorem 3.12 and Gauss–Bonnet's theorem. □

If $\tilde{M}$ is diffeomorphic to a two-sphere, theorem 3.14 yields the following

Corollary 3.15. *Let $\tilde{M}$ be a Kähler manifold which is diffeomorphic to a two-sphere S^2. Then an imbedded closed curve L in $\tilde{M}$ is harmonic minimal if and only if L divides $\tilde{M}$ into two regions D_1, D_2 with equal total Gauss curvature, i.e., $\int_{D_1} G\, dv = \int_{D_2} G\, dv$, where G is the Gauss curvature of $\tilde{M}$.*

When $\tilde{M}$ is a two-sphere S^2 with the standard Kähler structure, then an imbedded closed curve L in $\tilde{M}$ is harmonic minimal if and only if L is area bisecting.

For closed curves in a nonpositively curved Kähler surface, we have the following

Corollary 3.16. *If $\tilde{M}$ is a Kähler surface with nonpositive Gauss curvature, then every harmonic minimal closed curve in $\tilde{M}$ has self-intersection points.*

Proof. This follows immediately from theorem 3.14. □

Another application of theorem 3.12 is the following result, which provides us with many examples of harmonic minimal submanifolds.

Proposition 3.17. *Let L_i be closed Lagrangian curves of some Kähler manifolds $\tilde{M}_i$, $i=1, \ldots, n$, respectively. Then the product Lagrangian submanifold $L = L_1 \times \cdots \times L_n$ is harmonic minimal in $\tilde{M} = \tilde{M}_1 \times \cdots \times \tilde{M}_n$ if and only if L_i are harmonic minimal in $\tilde{M}_i$, $i=1, \ldots, n$.*

Proof. Let $e_1, \ldots, e_n$ be a field of orthonormal frames on the product Lagrangian submanifold L such that each e_i is tangent to L_i. Denote by $\omega^1, \ldots, \omega^n$ the field of dual frames. Then

$$\alpha_H = -\frac{1}{n}\{\kappa_1 \omega^1 + \cdots + \kappa_n \omega^n\}, \tag{3.7}$$

where κ_i is the curvature of the Lagrangian curves L_i in $\tilde{M}_i$. Since L is the Riemannian product of $L_1, \ldots, L_n$, $\omega^1, \ldots, \omega^n$ are harmonic one-forms on L. From (3.7) and theorem 3.12 it follows that $(\alpha_H, \omega^i) = 0$ if and only if L_i is harmonic minimal in $\tilde{M}_i$. Thus, by applying proposition 3.5, we obtain proposition 3.17. □

Remark 3.18. Proposition 3.17 implies, for instance, that there exist infinitely many harmonic minimal Lagrangian surfaces in the compact Hermitian symmetric space $Q_2 = SO(4)/SO(2) \times SO(2)$ which is the Riemannian product of two complex projective lines. For harmonic stability of these harmonic minimal surfaces, see section 5.

3.5. ISOTROPIC SURFACES

For E-minimal Lagrangian surfaces we have the following

Proposition 3.19. *Let L be a Lagrangian E-minimal surface in an Einstein–Kähler manifold $\tilde{M}$ which is not minimal in the usual sense. Then*

(1) If L has constant mean curvature in $\tilde{M}$, then L is flat;

(2) Conversely, if L is flat and compact, then L has constant nonzero mean curvature in $\tilde{M}$.

In order to prove this proposition we give the following lemma.

Lemma 3.20. *Let I be an isotropic surface of a Kähler manifold $\tilde{M}$ with $d\alpha_H = 0$. If I is E-minimal, then*

(1) either α_H vanishes identically or α_H has only isolated zeros;

(2) the Gaussian curvature G of I satisfies

$$G = -\tfrac{1}{2}\Delta \ln \| \alpha_H \|^2, \tag{3.8}$$

on the open subset of I on which $H \neq 0$.

Proof. Under the hypothesis of the lemma, we obtain from corollary 3.6 that

$$d\alpha_H = \delta\alpha_H = 0 \ . \tag{3.9}$$

Let (x, y) be an isothermal coordinate system of I. We put $\alpha_H = p\,dx + q\,dy$. Then (3.9) implies that the functions $p(x, y)$ and $q(x, y)$ satisfy the following Cauchy–Riemann condition:

$$\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}, \qquad \frac{\partial p}{\partial x} = -\frac{\partial q}{\partial y} .$$

From this we conclude that $p(x, y) + iq(x, y)$ is a holomorphic function in $z = x + iy$. Consequently, we have statement (1).

For statement (2), we choose a local field of orthonormal frames such that e_1 is parallel to JH. So, we may put $H = he_{1*} = hJe_1$ and hence lemma 2.1 implies

$$\langle \nabla^{\perp}_{e_1}(he_{1*}), e_{2*} \rangle = \langle \nabla^{\perp}_{e_2}(he_{1*}), e_{1*} \rangle \ .$$

Combining this with (2.4) we obtain

$$e_2 h = h\omega_1^2(e_1) \ . \tag{3.10}$$

If I is E-minimal, then, by proposition 3.4, we have $\delta\alpha_H = 0$. Thus, by applying lemma 3.6, we may obtain

$$e_1 h = -h\omega_1^2(e_2) \ . \tag{3.11}$$

Since $\|\alpha_H\|^2 = h^2$, (3.10) and (3.11) imply

$$2\omega_1^2 = (e_2(\ln \|\alpha_H\|^2))\omega^1 - (e_1(\ln \|\alpha_H\|^2))\omega^2 \ . \tag{3.12}$$

By taking the exterior derivative of (3.12) and applying structure equations together with (3.10) and (3.11), we may find

$$2d\omega_1^2 = (\Delta \ln \|\alpha_H\|^2)\omega^1 \wedge \omega^2 \ . \tag{3.13}$$

On the other hand, it is well known that

$$d\omega_1^2 = -G\omega^1 \wedge \omega^2 \ . \tag{3.14}$$

Thus, by combining (3.13) and (3.14), we obtain (3.8). This proves lemma 3.20. □

Proof of proposition 3.19. Now, if L is a compact Lagrangian surface in an Einstein–Kähler manifold $\tilde{M}$ and if L is E-minimal, then the hypothesis of lemma 3.20 holds automatically. Suppose L is flat and it is not minimal in the usual sense, then lemma 3.20 implies

$$\Delta \ln \|\alpha_H\|^2 = 0 \ . \tag{3.15}$$

Because $\|\alpha_H\|^2$ is a nonnegative differentiable function and L is compact, (3.15) implies that $\|\alpha_H\|$ is a positive constant. Thus, L has constant nonzero mean curvature. Conversely, if the Lagrangian surface has constant nonzero mean curvature, then (3.8) implies that L is flat. □

Corollary 3.21. *Let L be a Lagrangian surface in $\mathbb{C}^2$. If L is E-minimal with constant mean curvature, then L is an open portion of one of the following surfaces: (a) a minimal Lagrangian surface of $\mathbb{C}^2$; (b) the product of two circles; or (c) the product of a circle and a line.*

Proof. Let L be a Lagrangian surface in $\mathbb{C}^2$. Assume L is E-minimal and L has constant mean curvature. Then either L is minimal in the usual sense or L is flat. If the latter case occurs, L is an open portion of the product of two circles or an open portion of the product of an open portion of a line and a circle. □

Remark 3.22. Let L be a Lagrangian surface in $\mathbb{C}^2$ and let $\nu: L \to \mathrm{G}(2,4)$ be its associated Gauss map from L into the real Grassmannian $\mathrm{G}(2,4)$. Since $\mathrm{G}(2,4)$ is isometric to the product of two spheres: $\mathrm{G}(2,4) \cong \mathrm{S}^2_- \times \mathrm{S}^2_+$, there is a canonical decomposition of the Gauss map: $\nu = \nu_1 \times \nu_2$ (cf., for instance, ref. [4]). By corollary 3.7, we may conclude that a Lagrangian surface in $\mathbb{C}^2$ is E-minimal if and only if $\nu_2: L \to \mathrm{S}^2_-$ is a harmonic map.

4. Second variational formulas for isotropic submanifolds

For isotropic submanifolds in a Kähler manifold, we have the following result, which generalizes the second variational formulas of Lagrangian submanifolds obtained in refs. [3,12].

Proposition 4.1. *Let I be a compact n-dimensional isotropic submanifold of a Kähler manifold $\tilde{M}$. Suppose that η is a vector field on $\tilde{M}$ such that $\eta|_I$ is normal to I and let ϕ_t denote the flow generated by η in a neighbourhood of I in $\tilde{M}$. Then we have:*

(i) If $\eta|_I \in \Gamma(J(TI))$, then

$$\mathscr{V}''(\eta) = \int_I \Big(\tfrac{1}{2}\|d\alpha_\eta\|^2 + \|\delta\alpha_\eta\|^2 + 2\|\mathrm{proj}_\nu \sigma(X, \cdot)\|^2 - n\langle H, \sigma(X,X)\rangle + n^2\langle H, \eta\rangle^2 - n\langle H, \tilde{\nabla}_\eta \eta\rangle - \sum_{i=1}^{n} [\tilde{R}(e_i, \eta; \eta, e_i) + \tilde{R}(Je_i, \eta; \eta, Je_i)] \Big) \mathrm{d}v, \qquad (4.1)$$

where $e_1, \ldots, e_n$ is an orthonormal frame of I, $X=-J\eta$, and ω is the dual one-form of $J\eta|_I$.

(ii) If $\eta|_I \in \Gamma(\nu)$, then

$$\mathscr{V}''(\eta)=\int_I \Big(\|A_{J\eta}\|^2-\|A_\eta\|^2+\|\mathrm{proj}_\nu \nabla^\perp \eta\|^2$$

$$+n^2\langle H,\eta\rangle^2-n\langle H,\tilde{\nabla}_\eta \eta\rangle-\sum_{i=1}^n \tilde{R}(e_i,\eta;\eta,e_i)\Big)\,\mathrm{d}v\,, \tag{4.2}$$

where $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal frame of I.

Proof.

(i) If $\eta|_I \in (J(TI))$, we can find a tangent vector field $X \in \Gamma(TI)$ such that $JX=\eta|_I$. Using the fact that $\tilde{M}$ is Kählerian, we have, for $\eta \in J(TI)$,

$$\langle \nabla^\perp \eta, \zeta\rangle=-\langle \nabla X, J\zeta\rangle\,, \qquad \mathrm{proj}_\nu \nabla^\perp \eta=J(\mathrm{proj}_\nu \sigma(X,\cdot))\,,$$

from which we find

$$\|\nabla^\perp \eta\|^2=\|\nabla X\|^2+\|\mathrm{proj}_\nu \nabla^\perp \eta\|^2=\|\nabla X\|^2+\|\mathrm{proj}_\nu \sigma(X,\cdot)\|^2\,. \tag{4.3}$$

On the other hand, the Gauss equation gives

$$\sum_{i=1}^n \tilde{R}(e_i,\eta,\eta,e_i)=\sum_{i=1}^n \tilde{R}(Je_i,X,X,Je_i)$$

$$=\tilde{S}(X,X)-\sum_{i=1}^n \tilde{R}(e_i,X,X,e_i)-\sum_{j=1}^{2p} \tilde{R}(\epsilon_j,X,X,\epsilon_j)$$

$$=\tilde{S}(X,X)-S(X,X)+n\langle H,\sigma(X,X)\rangle$$

$$-\sum_{i=1}^n \|\sigma(X,e_i)\|^2-\sum_{j=1}^{2p} \tilde{R}(\epsilon_j,X,X,\epsilon_j)\,,$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal frame of TI and $\{e_1, \ldots, e_{2p}\}$ is an orthonormal frame of the maximal complex normal subbundle ν. We put

$$\sum_{i=1}^n \|\sigma(X,e_i)\|^2=\|\sigma(X,\cdot)\|^2\,.$$

Thus, from lemma 3.1 we get

$$\mathscr{V}''(\eta)=\int_I \Big(\|\nabla X\|^2+\|\mathrm{proj}_\nu \sigma(X,\cdot)\|^2-\tilde{S}(X,X)+S(X,X)$$

$$-n\langle H,\sigma(X,X)\rangle+\|\sigma(X,\cdot)\|^2+\sum_{i=1}^{2p} \tilde{R}(\epsilon_j,X,X,\epsilon_j)$$

$$+n^2\langle H,\eta\rangle^2-n\langle H,\tilde{\nabla}_\eta \eta\rangle-\|A_\eta\|^2\Big)\,\mathrm{d}v\,. \tag{4.4}$$

Since

$$\|\sigma(X,\cdot)\|^2-\|A_\eta\|^2=\|\mathrm{proj}_\nu\,\sigma(X,\cdot)\|^2\,,$$

we have

$$\mathscr{V}''(\eta)=\int_I\Big(\|\nabla X\|^2+2\|\mathrm{proj}_\nu\,\sigma(X,\cdot)\|^2-\tilde{S}(X,X)+S(X,X)$$

$$-n\langle H,\sigma(X,X)\rangle+\sum_{i=1}^{2p}\tilde{R}(\epsilon_j,X,X,\epsilon_j)$$

$$+n^2\langle H,\xi\rangle^2-n\langle H,\tilde{\nabla}_\eta\eta\rangle\Big)\,\mathrm{d}v\,.$$

Finally, put $W=\nabla_X X+(\mathrm{div}\,X)X$, where $JX=\eta|_I$, and denote by div X the divergence of X. Then, by computing the divergence of W, we have

$$\begin{aligned}0&=\int_I(\mathrm{div}\,W)\,\mathrm{d}v\\&=\int_I[S(X,X)+\|\nabla X\|^2-(\tfrac{1}{2}\|d\alpha_\eta\|^2+\|\delta\alpha_\eta\|^2)]\,\mathrm{d}v\,.\end{aligned}\tag{4.5}$$

Consequently, by (4.4) and (4.5), we obtain (4.1).

(ii) If $\eta|_I\in\Gamma(\nu)$, then, for any vector fields Y, Z tangent to I, we have

$$\langle\nabla_Y^\perp\eta,JZ\rangle=-\langle\tilde{\nabla}_X J\eta,Z\rangle=\langle A_{J\eta}Y,Z\rangle\,.\tag{4.6}$$

This implies

$$\|\nabla^\perp\eta\|^2=\|A_{J\eta}\|^2+\|\mathrm{proj}_\nu\,\nabla^\perp\eta\|^2\,.\tag{4.7}$$

Thus, from lemma 3.1, we obtain (4.2). □

Corollary 4.2 *Let I be a compact n-dimensional isotropic submanifold of a Kähler manifold $\tilde{M}$. Suppose that η is a vector field on $\tilde{M}$ such that $\eta|_I$ is normal to I and let ϕ_t denote the flow generated by η in a neighborhood of I in $\tilde{M}$. If I is minimal in the usual sense, then:*

(i) If $\eta|_I\in\Gamma(J(TI))$, then

$$\mathscr{V}''(\eta)=\int_I\Big(\tfrac{1}{2}\|d\alpha_\eta\|^2+\|\delta\alpha_\eta\|^2+2\|\mathrm{proj}_\nu\,\sigma(J\eta,\cdot)\|^2$$

$$-\sum_{i=1}^{n}[\tilde{R}(e_i,\eta;\eta,e_i)+\tilde{R}(Je_i,\eta;\eta,Je_i)]\Big)\,\mathrm{d}v\,,\tag{4.8}$$

where $e_1, ..., e_n$ is an orthonormal frame of I and ω is the dual one-form of $J\xi$.
(ii) If $\eta|_I \in \Gamma(\nu)$, then

$$\mathscr{V}''(\eta)=\int_I\left(\|A_{J\eta}\|^2-\|A_\eta\|^2+\|\mathrm{proj}_\nu\,\nabla^\perp\eta\|^2-\sum_{i=1}^n\tilde{R}(e_i,\eta;\eta,e_i)\right)\mathrm{d}v\,, \quad (4.9)$$

where $\{e_1, e_2, ..., e_n\}$ is an orthonormal frame of I.

This corollary follows immediately from proposition 4.1.

For isotropic E-minimal submanifolds, we have the following second variational formulas under exact variations.

Corollary 4.3. *Let I be a compact n-dimensional isotropic E-minimal submanifold of a Kähler manifold $\tilde{M}$. Then:*
(i) For $\xi=JX\in\Gamma(J(TI))\cap\mathscr{E}$, we have

$$\mathscr{V}''(\xi)=\int_I\Bigg(\tfrac{1}{2}\|d\alpha_\xi\|^2+\|\delta\alpha_\xi\|^2+2\|\mathrm{proj}_\nu\,\sigma(X,\cdot)\|^2$$

$$-\sum_{i=1}^n[\tilde{R}(e_i,\xi;\xi,e_i)+\tilde{R}(Je_i,\xi;\xi,Je_i)]$$

$$+n^2\langle H,\xi\rangle^2-n\langle H,\sigma(X,X)\rangle\Bigg)\mathrm{d}v\,, \quad (4.10)$$

where $e_1, ..., e_n$ is an orthonormal frame of I.
(ii) For $\xi\in\Gamma(\nu)\cap\mathscr{E}$, we have

$$\mathscr{V}''=\int_I\left(\|A_{J\xi}\|^2-\|A_\xi\|^2+\|\mathrm{proj}_\nu\,\nabla^\perp\xi\|^2-\sum_{i=1}^n\tilde{R}(e_i,\xi;\xi,e_i)\right)\mathrm{d}v\,. \quad (4.11)$$

Proof. This corollary follows from proposition 4.1 and the fact that, for any $\xi\in\mathscr{E}$, we have $\nabla_\xi\xi=0$ (cf. section 3). □

Similarly, for isotropic harmonic minimal submanifolds, we have the following second variational formulas under harmonic variations.

Corollary 4.4. *Let I be a compact n-dimensional isotropic harmonic minimal submanifold of a Kähler manifold $\tilde{M}$. Then:*
(i) For $\xi=JX\in\Gamma(J(TI))\cap\mathscr{H}$, we have

$$\mathscr{V}''(\xi) = \int_I \Big(2\|\mathrm{proj}_\nu\, \sigma(X, \cdot)\|^2 - \sum_{i=1}^{n} [\tilde{R}(e_i, \xi; \xi, e_i) + \tilde{R}(Je_i, \xi; \xi, Je_i)] + n^2\langle H, \xi\rangle^2 - 2\langle H, \sigma(X, X)\rangle \Big) \mathrm{d}v\,, \tag{4.12}$$

where $e_1, \ldots, e_n$ *is an orthonormal frame of* I.

(ii) For $\xi \in \Gamma(\nu) \cap \mathscr{H}$, *we have*

$$\mathscr{V}''(\xi) = \int_I \Big(\|A_{J\xi}\|^2 - \|A_\xi\|^2 + \|\mathrm{proj}_\nu\, \nabla^\perp \xi\|^2 - \sum_{i=1}^{n} \tilde{R}(e_i, \xi; \xi, e_i) \Big) \mathrm{d}v\,. \tag{4.13}$$

5. Some applications

By applying proposition 4.1, we obtain the following

Theorem 5.1. *Let* I *be a compact isotropic submanifold of a Kähler manifold* $\tilde{M}$ *with nonpositive holomorphic bisectional curvatures. If* I *is minimal in the usual sense, then*

(1) I *is stable under deformations with respect to* $\Gamma(J(TI))$;

(2) $\mathscr{V}''(\xi) + \mathscr{V}''(J\xi) \geq 0$ *for any* $\xi \in \Gamma(\nu)$.

Proof. Follows easily from corollary 4.2. □

The following example shows that theorem 5.1 is the best possible.

Example 5.2. Let I be a compact, non totally geodesic, minimal hypersurface of an $(n+1)$-dimensional flat real torus RT^{n+1} which is imbedded in a flat complex torus CT^{n+1} as a totally geodesic, Lagrangian submanifold. Then I is an isotropic submanifold of CT^{n+1}. Let ξ be a unit normal vector field of I in RT^{n+1}. Then we have

$$A_\xi \neq 0\,, \qquad \nabla^\perp \xi = \nabla^\perp J\xi = 0\,, \qquad A_{J\xi} = 0\,.$$

Since CT^{n+1} is flat, we obtain

$$\mathscr{V}''(\xi) < 0\,, \qquad \mathscr{V}''(J\xi) > 0\,.$$

In ref. [10], Takeuchi proved the following

Theorem 5.3. *Let $\tilde{M}$ be a Hermitian symmetric space of compact type and M a compact totally geodesic Lagrangian submanifold of $\tilde{M}$. Then M is a stable submanifold in the usual sense if and only if M is simply connected.*

In view of theorem 5.3, we give the following

Theorem 5.4. *Let M be either a totally geodesic Lagrangian submanifold or a totally geodesic isotropic submanifold of a Kähler manifold $\tilde{M}$ and I a compact minimal hypersurface of M. If either (a) the sectional curvature of $\tilde{M}$ is nonnegative and I is not totally geodesic, or (b) the sectional curvature of $\tilde{M}$ is positive, then I is E-unstable in $\tilde{M}$ and also harmonic unstable in $\tilde{M}$. In particular, it is unstable in the usual sense.*

Proof. Let M be a totally geodesic Lagrangian (or isotropic) submanifold of a Kähler manifold and I a compact minimal hypersurface of M. Let ξ be the unit normal vector field of I in M. Then $\xi \in \Gamma(\nu) \cap \mathscr{H}$. If I is not totally geodesic in M and M is totally geodesic in $\tilde{M}$, then we have

$$A_\xi \neq 0 , \qquad A_{J\xi} = 0 .$$

Furthermore, since I is a hypersurface of M and M is totally geodesic in $\tilde{M}$, we also have

$$\text{proj}_\nu \nabla^\perp \xi = 0 .$$

Therefore, by applying corollary 4.3 we conclude that $\mathscr{V}''(\xi) < 0$. Thus, I is both E-unstable and harmonic unstable in $\tilde{M}$. In particular, this shows that I is unstable in the usual sense. The other case can be proved in a similar way. □

Theorem 5.4 can be applied in particular to the case where $\tilde{M}$ is a compact symmetric space. A submanifold $\tilde{M}$ of a Kähler manifold $\tilde{M}$ is called a *real form* of $\tilde{M}$ if M is the fixed point set of an involutive anti-holomorphic isometry of $\tilde{M}$. It is known that the real forms of a Kähler manifold are totally geodesic submanifolds which are either isotropic or Lagrangian in $\tilde{M}$ (cf. ref. [13]).

For real forms of compact Hermitian symmetric spaces, we have the following

Corollary 5.5. *Let $\tilde{M}$ be a Hermitian symmetric space of compact type and M a real form of $\tilde{M}$. Assume I is a compact minimal hypersurface of M.*

(i) If I is not totally geodesic in M, then I is E-unstable and harmonic unstable in $\tilde{M}$.

(ii) If $\tilde{M}$ is irreducible and $\dim M > \operatorname{rank}(\tilde{M})$, *then I is E-unstable and harmonic unstable in $\tilde{M}$.*

Proof. Statement (i) follows from theorem 5.4. For statement (ii), if $\tilde{M}$ is a Hermitian symmetric space of compact type with $\dim M > \operatorname{rank}(\tilde{M})$, then $\mathscr{V}''(\xi) < 0$, where ξ is the unit normal vector field of I in M. Thus, by applying a similar argument as in the proof of theorem 5.3, we conclude that I is E-unstable and harmonic unstable. □

It has been proved in ref. [11] that the canonical totally geodesic RP^n in CP^n is Hamiltonian stable. However, corollary 5.4 implies the following

Corollary 5.6. *Let* RP^{n+1} *be a real projective space canonically imbedded in a complex projective space* CP^{n+1} *(with the standard Fubini–Study metric) as a totally geodesic Lagrangian submanifold. Then every compact minimal hypersurface M of* RP^{n+1} *is E-unstable in* CP^{n+1} *and also harmonic unstable.*

Now we study the stability of harmonic minimal product Lagrangian submanifolds.

Proposition 5.7. *Let $L_1, \dots, L_n$ be n closed, harmonic minimal, Lagrangian curves of the Kähler manifolds $\tilde{M}_1, \dots, \tilde{M}_n$, respectively. Then the product $L = L_1 \times \cdots \times L_n$ is a harmonic minimal Lagrangian submanifold in the Kähler manifold $\tilde{M} = \tilde{M}_1 \times \cdots \times \tilde{M}_n$. Moreover, for any harmonic normal variation $\xi \in \mathscr{H}$, we have*

$$\mathscr{V}''(\xi) = -\int_L \tilde{S}(\xi, \xi)\, dv. \tag{5.1}$$

Proof. Since L_i is harmonic minimal in $\tilde{M}_i$ for each i, $L = L_1 \times \cdots \times L_n$ is harmonic minimal in $\tilde{M} = \tilde{M}_1 \times \cdots \times \tilde{M}_n$ by proposition 3.17. Let $e_1, \dots, e_n$ be unit tangent vector fields of $L_1, \dots, L_n$, respectively. Then $e_1, \dots, e_n$ can be regarded as a field of globally defined orthonormal frames on L. Denote by $\omega^1, \dots, \omega^n$ the field of dual one-forms of $e_1, \dots, e_n$ on L. Then $\omega^1, \dots, \omega^n$ form an orthonormal basis of the space of harmonic one-forms on L. Thus, every harmonic normal vector field ξ of L is a linear combination of $Je_1, \dots, Je_n$. We denote the unique normal harmonic variation associated with ξ also by ξ as before.

Let u_i denote an arc length parametrization of L_i and $\kappa_i(u_i)$ the curvature function of L_i in $\tilde{M}_i$. Then, for any harmonic normal vector field

$$\xi = \sum_i c_i J e_i\,, \tag{5.2}$$

we have

$$\sigma(X,X)=\sum_i^n c_i^2\kappa_i Je_i\,,\qquad H=\frac{1}{n}\sum_i^n \kappa_i Je_i\,,\tag{5.3}$$

where $X=J\xi$. From (5.2) and (5.3), we obtain

$$n^2\langle H,\xi\rangle^2-n\langle H,\sigma(X,X)\rangle=2\sum_{i<j}c_ic_j\kappa_i\kappa_j\,.\tag{5.4}$$

Hence we obtain

$$\int_L [n^2\langle H,\xi\rangle^2-n\langle H,\sigma(X,X)\rangle]\,\mathrm{d}v$$

$$=2\sum_{i<j}c_ic_j\frac{\mathrm{vol}(L)}{\mathrm{length}(L_i)\ \mathrm{length}(L_j)}\int_{L_i}\kappa_i(u_i)\,\mathrm{d}u_i\int_{L_j}\kappa_j(u_j)\,\mathrm{d}u_j=0\,,$$

by virtue of theorem 3.12. Combining this with corollary 4.4, we obtain (5.1). □

From proposition 5.7 we obtain immediately the following stability result.

Proposition 5.8. *Let $L_1, \ldots, L_n$ be n closed, harmonic minimal, Lagrangian curves of Kähler-manifolds $\tilde{M}_1, \ldots, \tilde{M}_n$, respectively. Then*

(1) if $\tilde{M}_1, \ldots, \tilde{M}_n$ are nonnegatively curved and at least one of $\tilde{M}_1, \ldots, \tilde{M}_n$ is positively curved, then the product $L=L_1\times\cdots\times L_n$ is a harmonic minimal Lagrangian submanifold which is harmonic unstable in the Kähler manifold $\tilde{M}=\tilde{M}_1\times\cdots\times\tilde{M}_n$;

(2) if each $\tilde{M}_i$ is the complex Euclidean line $\mathbb{C}^1$, then the volume of the product Lagrangian submanifold $L=L_1\times\cdots\times L_n$ is invariant under harmonic variations; and

(3) if $\tilde{M}_1, \ldots, \tilde{M}_n$ are nonpositively curved, then the product $L=L_1\times\cdots\times L_n$ is a harmonic Lagrangian submanifold which is harmonic stable in the Kähler manifold $\tilde{M}=\tilde{M}_1\times\cdots\times\tilde{M}_n$.

In particular, proposition 5.8 yields the following corollaries.

Corollary 5.9. *Let L be a closed, harmonic minimal, Lagrangian curve in a Kähler manifold $\tilde{M}$. Then*

(1) if $\tilde{M}$ is positively curved, L is harmonic unstable;

(2) if $\tilde{M}$ is the complex Euclidean line $\mathbb{C}^1$, then the length of L is invariant under harmonic variations; and

(3) if $\tilde{M}$ is nonpositively curved, L is harmonic stable.

Corollary 5.10. *Let $Q_2 = CP^1 \times CP^1$ be the complex quadric which is the Riemannian product of two complex projective lines and let L_i, $i=1, 2$ be two closed, harmonic minimal curves in CP^1. Then the product $L = L_1 \times L_2$ is a harmonic minimal, Lagrangian surface which is harmonic unstable in Q_2.*

Remark 5.11. We remark that corollary 4.4 implies every compact, minimal, Lagrangian submanifold of a Kähler manifold with positive first Chern form is harmonic unstable whenever it is defined.

Remark 5.12. Finally, in view of the results in this article, we propose here the following problems:

Problem 1. *Classify the E-minimal and harmonic minimal isotropic submanifolds of $\mathbb{C}^m$.*

Problem 2. *Classify the stable E-minimal and stable harmonic minimal isotropic submanifolds of $\mathbb{C}^m$.*

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